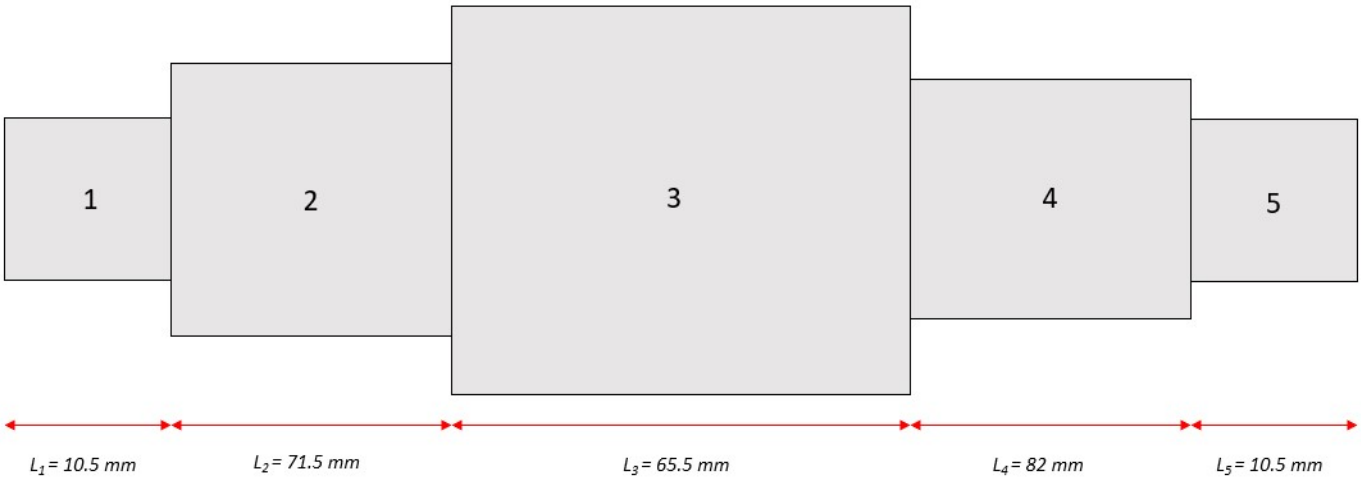
Problem: Verify that the deflection at each gear is less than 0.127 mm.

Before calculating the deflection in the shaft, we find the support forces acting on the bearings of the shaft. Then we will use singularity functions to solve for deflection assuming a constant, minimum cross-section.

Known Parameters:

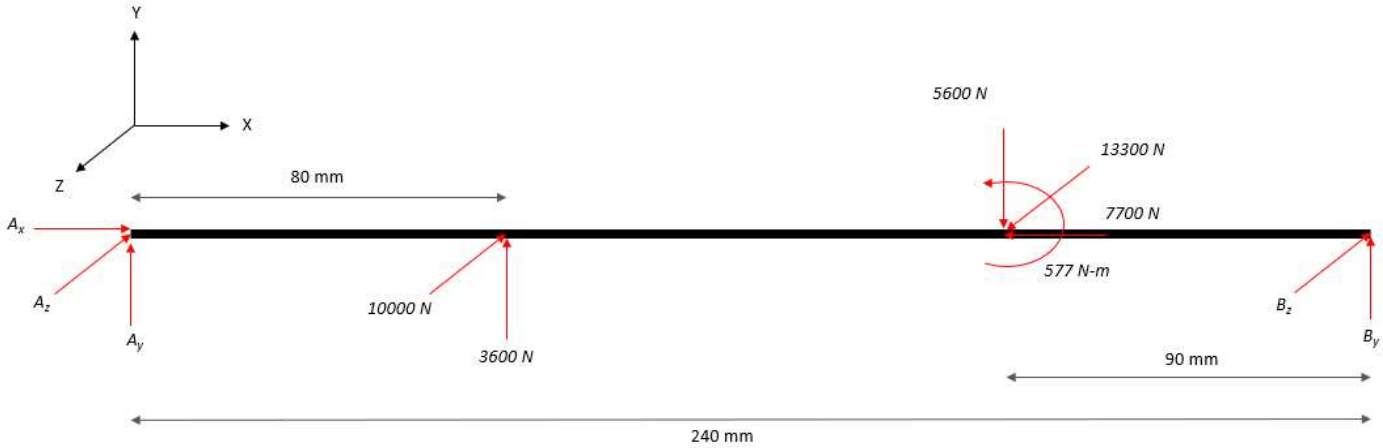


Note: The actual length of sections 1 and 5 are 21 mm each; however, the bearing support forces act 10.5 mm from either end. For simplicity, we will assume the bearing support reactions are acting at the ends of the beam- thus, sections 1 and 5 have been halved in length.

Section	**1**	**2**	**3**	**4**	**5**
d (mm)	35	45	Stock size (57.15)	50	35
I (10^{-8} m^4)	7.37	20.1	52.4	30.7	7.37
J (10^{-8} m^4)	14.7	40.3	105	61.4	14.7
Start position (x_i mm)	0	10.5	82	147.5	229.5
End position (x_f mm)	10.5	82	147.5	229.5	240
Segment length (m)	0.0105	0.0715	0.0655	0.082	0.0105

We will be using the smallest diameter of the shaft to solve for deflection which is **35 mm** in this case. Thus, if the shaft with a minimum cross section meets all the conditions, we can be sure the stepped shaft with larger cross-sections will too.

Free Body Diagram:



Assumptions:

1. The beam is made of a homogeneous material (has a constant modulus of elasticity across its length).
2. The material is in the linear elastic region (Hooke's Law is applicable).
3. Deformed angels and displacements are small.
4. The longitudinal axis, which acts within the neutral surface, does not experience any change in length.
5. All cross-sections of the beam remain plane and perpendicular to the longitudinal axis during deformation.

Equlibrium Equations:

1. $\Sigma F_x := 0$

   $A_x := 7700\ \mathrm{N}$
2. $\Sigma F_y := 0$

   $\Sigma F_y := A_y + 3600 - 5600 + B_y$

   $\Sigma F_y := -2000 + A_y + B_y$
3. $\Sigma M_z := 0$

   $\Sigma M_z := B_y \cdot 0.24 + 3600 \cdot 0.08 - 5600 \cdot 0.15 + 577$

   $B_y := -104.17\ \mathrm{N}$

   $A_y := 2104.17\ \mathrm{N}$
4. $\Sigma F_z := 0$

   $\Sigma F_z := A_z - 10000 + 13300 + B_z$

   $\Sigma F_z := A_z + B_z + 3300$
5. $\Sigma My := 0$

   $\Sigma My := (-10000) \cdot 0.08 + B_z \cdot 0.24 + 13300 \cdot 0.15$

   $B_z := -4979.17\ \mathrm{N}$

   $A_z := 1679.17\ \mathrm{N}$

Singularity Functions:

In this section, we will solve for deflection in the stepped shaft by assuming the diameter of the entire shaft is 35mm as this is the smallest diameter of the stepped shaft. If the deflection in both gears is less than 0.127 mm using the smallest diameter of the shaft, then we can be sure that the larger diameter also has a deflection less than 0.127 mm.

The singularity functions for My and Mz can be written as shown below:

$$M_y(x) := A_z \cdot x^1 - 10000 \cdot (x - 0.08)^1 + 13300 \cdot (x - 0.15)^1 + B_z \cdot (x - 0.24)^1 \quad (1)$$

$$M_z(x) := A_y \cdot x^1 + 3600 \cdot (x - 0.08)^1 - 5600 \cdot (x - 0.15)^1 - 577 \cdot (x - 0.15)^0 + B_y \cdot (x - 0.24)^1 \quad (2)$$

We can use the following formula to determine deflection y and z components of deflection in the beam:

$$y(x) := \int_0^x \int_0^x \frac{M_z}{E \cdot I} \, dx \, dx \quad (3)$$

$$z(x) := \int_0^x \int_0^x \frac{M_y}{E \cdot I} \, dx \, dx \quad (4)$$

Therefore, by substituing equations (2) and (1) into (3) and (4) respectively and integrating twice, we can derive the following equations:

$$y(x) := \frac{1}{E \cdot I} \cdot \left(\frac{A_y \cdot x^3}{6} + \frac{3600 \cdot (x - 0.08)^3}{6} - \frac{5600 \cdot (x - 0.15)^3}{6} - \frac{577 \cdot (x - 0.15)^2}{2} + \frac{B_y \cdot (x - 0.24)^3}{6} + C_1 \cdot x + C_2 \right) \quad (5)$$

$$z(x) := \frac{1}{E \cdot I} \cdot \left(\frac{A_z \cdot x^3}{6} - \frac{10000 \cdot (x - 0.08)^3}{6} + \frac{13300 \cdot (x - 0.15)^3}{6} + \frac{B_z \cdot (x - 0.24)^3}{6} + C_3 \cdot x + C_4 \right) \quad (6)$$

Finding Integration Constants

Before we can solve for deflection in the y and z direction, we first need to solve for the integration constants by writing boundary conditions for the shaft. Since both bearings restrict deflection in the y and z directions, we know that the y and z components of deflection must be 0. Therefore, the following boundary conditions can be written:

$$y(0) := 0 \quad (7)$$

$$z(0) := 0 \quad (8)$$

$$y(0.24) := 0 \quad (9)$$

$$z(0.24) := 0 \quad (10)$$

By substituting equations (7) and (9) into equation (5) and equations (8) and (10) into (6), we can solve for the integration constants for both deflection equations:

Recall:

$E := 200 \text{ GPa}$

$I := 0.0000000737 \text{ m}^4$

Using Boundary Condition (7): $x := 0 \quad y(x) := 0$

$$y(x) := \frac{1}{E \cdot I} \cdot (0 + 0 + 0 + 0 + 0 + 0 + C_2)$$

$C_2 := 0$

Using Boundary Condition (9): $x := 0.24 \quad y(x) := 0$

$$y(x) := \frac{1}{E \cdot I} \cdot \left(\frac{A_y \cdot 0.24^3}{6} + \frac{3600 \cdot (0.24 - 0.08)^3}{6} - \frac{5600 \cdot (0.24 - 0.15)^3}{6} - \frac{577 \cdot (0.24 - 0.15)^2}{2} + \frac{B_y \cdot (0.24 - 0.24)^3}{6} + C_1 \cdot 0.24 + 0 \right)$$

$$C_1 := \frac{-\left(\frac{A_y \cdot (0.24 \text{ m})^3}{6} + \frac{3600 \text{ N} \cdot (0.24 \text{ m} - 0.08 \text{ m})^3}{6} - \frac{5600 \text{ N} \cdot (0.24 \text{ m} - 0.15 \text{ m})^3}{6} - \frac{577 \text{ N m} \cdot (0.24 \text{ m} - 0.15 \text{ m})^2}{2} + \frac{B_y \cdot (0.24 \text{ m} - 0.24 \text{ m})^3}{6} \right)}{0.24 \text{ m}}$$

$C_1 = -17.8682\ \text{m J}$

Therefore :

$$y(x) := \frac{1}{E \cdot I} \cdot \left(\frac{A_y \cdot x^3}{6} + \frac{3600 \cdot (x - 0.08)^3}{6} - \frac{5600 \cdot (x - 0.15)^3}{6} - \frac{577 \cdot (x - 0.15)^2}{2} + \frac{B_y \cdot (x - 0.24)^3}{6} - 17.8682 \cdot x \right) \quad (11)$$

Using Boundary Condition (8): $x := 0$ $z(x) := 0$

$$z(x) := \frac{1}{E \cdot I} \cdot (0 + 0 + 0 + 0 + 0 + 0 + C_4)$$

$C_4 := 0$

Using Boudnary Condition (10): $x := 0.24$ $z(x) := 0$

$$z(x) := \frac{1}{E \cdot I} \cdot \left(\frac{A_z \cdot 0.24^3}{6} - \frac{10000 \cdot (0.24 - 0.08)^3}{6} + \frac{13300 \cdot (0.24 - 0.15)^3}{6} + \frac{B_z \cdot (0.24 - 0.24)^3}{6} + C_3 \cdot 0.24 + 0 \right)$$

$$C_3 := -\left(\frac{\frac{A_z \cdot (0.24\ \text{m})^3}{6} - \frac{10000\ \text{N} \cdot (0.24\ \text{m} - 0.08\ \text{m})^3}{6} + \frac{13300\ \text{N} \cdot (0.24\ \text{m} - 0.15\ \text{m})^3}{6} + \frac{B_z \cdot (0.24\ \text{m} - 0.24\ \text{m})^3}{6}}{0.24\ \text{m}} \right)$$

$C_3 = 5.5913\ \text{m J}$

Therefore :

$$z(x) := \frac{1}{E \cdot I} \cdot \left(\frac{A_z \cdot x^3}{6} - \frac{10000 \cdot (x - 0.08)^3}{6} + \frac{13300 \cdot (x - 0.15)^3}{6} + \frac{B_z \cdot (x - 0.24)^3}{6} + 5.5913 \cdot x \right) \quad (12)$$

Solving for Deflection:

Now we can solve for the y and z components of deflection at each gear. Recall:

Spur Gear Location:

$x_1 := 0.08\ \text{m}$

Using equations (11) and (12):

$$y(x_1) := \frac{1}{E \cdot I} \cdot \left(\frac{A_y \cdot x_1^3}{6} + \frac{3600\ \text{N} \cdot (x_1 - 0.08)^3}{6} - 17.8682\ \text{m J} \cdot x_1 \right)$$

$y(x_1) = -8.4796483 \cdot 10^{-5}\ \text{m}$

$$z(x_1) := \frac{1}{E \cdot I} \cdot \left(\frac{A_z \cdot x_1^3}{6} - \frac{10000\ \text{N} \cdot (x_1 - 0.08)^3}{6} + 5.5913\ \text{m J} \cdot x_1 \right)$$

$z\left(x_1\right) = 4.00673795 \cdot 10^{-5}\ \text{m}$

Helical Gear Location:

$x_2 := 0.15\ \text{m}$

$$y\left(x_2\right) := \frac{1}{E \cdot I} \cdot \left(\frac{A_y \cdot {x_2}^3}{6} + \frac{3600\ \text{N} \cdot \left(x_2 - 0.08\ \text{m}\right)^3}{6} - \frac{5600\ \text{N} \cdot \left(x_2 - 0.15\ \text{m}\right)^3}{6} - \frac{577\ \text{N m} \cdot \left(x_2 - 0.15\ \text{m}\right)^2}{2} - 17.8682\ \text{m J} \cdot x_2 \right)$$

$y\left(x_2\right) = -8.75735668 \cdot 10^{-5}\ \text{m}$

$$z\left(x_2\right) := \frac{1}{E \cdot I} \cdot \left(\frac{A_z \cdot {x_2}^3}{6} - \frac{10000\ \text{N} \cdot \left(x_2 - 0.08\ \text{m}\right)^3}{6} + \frac{13300\ \text{N} \cdot \left(x_2 - 0.15\ \text{m}\right)^3}{6} + 5.5913\ \text{m J} \cdot x_2 \right)$$

$z\left(x_2\right) = 0.0000822\ \text{m}$

Total Deflection:

The total deflection at any point x can be found using the following equation:

$$\delta := \sqrt{y\left(x\right)^2 + z\left(x\right)^2} \quad (13)$$

The deflection at the spur can be found as shown below:

$$\delta_1 := \sqrt{\left(\left(-8.4796483\right) \cdot 10^{-5}\ \text{m}\right)^2 + \left(4.00673795 \cdot 10^{-5}\ \text{m}\right)^2}$$

$\delta_1 = 9.37861313 \cdot 10^{-5}\ \text{m}$

The deflection at the helical gear:

$$\delta_2 := \sqrt{\left(-8.75735668 \cdot 10^{-5}\ \text{m}\right)^2 + \left(0.0000822\ \text{m}\right)^2}$$

$\delta_2 = 0.00012011\ \text{m}$

Conclusion :

Since the deflections at both gears is less than 0.000127 m, the **shaft deflection is acceptable**.